2017

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 1
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx.$ $\left.\frac{dV}{dt}\right\|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5,$ the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$V \approx [2(50.3) + 3(14.4) + 5(6.5)]$

$\approx 176.3 \text{ ft}^3$

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

The approximation in part a is an overestimate because A is a decreasing function.

Continue question 1 on page 5.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

$$V = \int_0^{10} \left(\frac{50.3}{e^{0.2h}+h}\right) dh$$

$$\approx 101.325 \text{ ft}^3$$

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$h = 5$ ft

$\frac{dh}{dt} = 0.26$ ft/min

$\frac{dV}{dt} = ?$

$$V = \int_0^{h} \left(\frac{50.3}{e^{0.2h}+h}\right) dh$$

$$\frac{dV}{dt} = \frac{50.3}{e^{0.2h}+h} \cdot \frac{dh}{dt}$$

$$= \frac{50.3}{e^{0.2(5)}+(5)} \cdot 0.26$$

$$\approx 1.694 \text{ ft}^3/\text{min}$$

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.

 1B₁

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$$V = [A(0)](2) + [A(2)](3) + [A(5)](5)$$
$$= (50.3)(2) + (14.4)(3) + (6.5)(5) = 176.3 \text{ ft}^3$$

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

The volume is overestimated, because the A(h) is continuosly decreasing, so after the value of A(h) at the left end of each subinterval, A(h) is lower for the rest of the subinterval (compared to the left end). which means the actual volume should be lower than the approximation.

Do not write beyond this border.

Continue question 1 on page 5.

 1B2

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

Volume equals to $\int_0^{10} f(h)\,dh \doteq 101 \text{ ft}^3$

(use of graphing calculator)

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$V = \int_0^5 f(h)\,dh$, $\frac{dV}{dt} = [f(5) - f(0)]\frac{dh}{dt}$

$\therefore \frac{dV}{dt} = (6.52 - 50.3)(0.26) = 11.3828 \text{ ft}^3/\text{minute}$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$2(14.4) + 3(6.5) + 5(2.9)$

$28.8 + 19.5 + 14.5 = 62.8 \text{ ft}^3$

The volume of the tank is the $\int_0^{10} A(h)$, therefore using a left Riemann sum, the volume of the tank is $\boxed{62.8 \text{ ft}^3}$.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

Because the function A is decreasing over the interval between $0 \leq h \leq 10$, the LRAM is an under-estimate of the volume in the tank.

Do not write beyond this border.

Continue question 1 on page 5.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

The Volume of the tank is the $\int f(h)\,dh$.

$$\int_0^{10} \frac{50.3}{e^{.2h}+h}\,dh = 101.325 \text{ ft}^3$$

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$h=5$

$\frac{dh}{dt} = .26$ ft/min

$\frac{dv}{dt} = f(h) = \frac{50.3}{e^{.2h}+h}$ when $h=5 = 6.517$ ft/min^2

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 1

Overview

In this problem students were presented with a tank that has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by a continuous and decreasing function A, where $A(h)$ is measured in square feet. Values of $A(h)$ for heights $h = 0,\ 2,\ 5,$ and 10 are supplied in a table. In part (a) students were asked to approximate the volume of the tank using a left Riemann sum and indicate the units of measure. Students needed to respond by incorporating data from the table in a left Riemann sum expression approximating $\int_0^{10} A(h)\,dh$ using the subintervals $[0, 2],\ [2, 5],$ and $[5, 10].$ [LO 3.2B/EK 3.2B2] In part (b) students needed to explain that a left Riemann sum approximation for the definite integral of a continuous, decreasing function overestimates the value of the integral. [LO 3.2B/EK 3.2B2] In part (c) the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$ is presented as a model for the area, in square feet, of the horizontal cross section at height h feet. Students were asked to find the volume of the tank using this model, again indicating units of measure. Using the model f for cross-sectional areas of the tank, students needed to express the volume of the tank as $\int_0^{10} f(h)\,dh$ and use the graphing calculator to produce a numeric value for this integral. [LO 3.4D/EK 3.4D2] In part (d) water is pumped into the tank so that the water's height is increasing at the rate of 0.26 foot per minute at the instant when the height of the water is 5 feet. Students were asked to use the model from part (c) to find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet, again indicating units of measure. Students needed to realize that the volume of water in the tank, as a function of its height h, is given by $V(h) = \int_0^h f(x)\,dx$ and then use the Fundamental Theorem of Calculus to find that the rate of change of the volume of water with respect to its height is given by $V'(h) = f(h).$ Then, using the chain rule for derivatives, students needed to relate the rates of change of volume with respect to time and height and the rate of change of the water's height with respect to time. Information in the problem suffices to be able to find these rates when the water's height is 5 feet. [LO 2.3C/EK 2.3C2, LO 3.3A/EK 3.3A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 1A
Score: 9

The response earned all 9 points: the units point, 2 points in part (a), 1 point in part (b), 2 points in part (c), and 3 points in part (d). The units point was earned because the units of ft^3 in parts (a) and (c) and ft^3/min in part (d) are all correct. In part (a) the left Riemann sum point was earned by the numerical expression in the first line. This expression would have also earned the approximation point without simplification. The student chooses to simplify, does so correctly, and thus earned the approximation point. In part (b) the statement "overestimate because A is a decreasing function" earned the point. In part (c) the definite integral earned the integral point, and 101.325 earned the answer point. In part (d) the second line on the right earned the 2 points for $\frac{dV}{dt}$. The third line on the right would have earned the answer point without simplification. The student chooses to give a final answer of 1.694 that is computed correctly and earned the answer point.

Question 1 (continued)

Sample: 1B
Score: 6

The response earned 6 points: the units point, 2 points in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). The units point was earned because the units of ft^3 in parts (a) and (c) and ft^3/minute in part (d) are all correct. In part (a) the left Riemann sum point was earned by the symbolic expression in the first line. The approximation point was earned by the second line. The numerical expression in the second line would have earned the approximation point without simplification to 176.3. The student chooses to simplify, does so correctly, and thus earned the approximation point. In part (b) the response of "overestimated" with the reason that "$A(h)$ is continuously decreasing" earned the point. In part (c) the expression $\int_0^{10} f(h)\, dh$ earned the integral point. The answer of 101 did not earn the answer point because the result is not accurate to three places after the decimal point. In part (d) the equation $\frac{dV}{dt} = [f(5) - f(0)]\frac{dh}{dt}$ earned 1 of the 2 $\frac{dV}{dt}$ points for using the chain rule. The student has made an error in the application of the Fundamental Theorem of Calculus. The numerical evaluation of the student's expression for $\frac{dV}{dt}$ is not eligible to earn the answer point.

Sample: 1C
Score: 3

The response earned 3 points: no units point, no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). The units of ft/min^2 in part (d) are incorrect, so the student did not earn the units point. Because the student uses a right Riemann sum, no points were earned in part (a). The confusion of left with right has implications in part (b). Because the setup in part (a) for a right Riemann sum is accurate, the student is eligible to earn the point in part (b) if the response is consistent for a right Riemann sum. In part (b) the implication that A is decreasing leads to an "under-estimate" is consistent with a right Riemann sum. Thus, the student earned the point. In part (c) $\int_0^{10} \frac{50.3}{e^{.2h} + h}\, dh$ earned the integral point, and 101.325 earned the answer point. In part (d) no points were earned because the statement of $\frac{dV}{dt} = f(h)$ is incorrect, and the remaining work only implies, incorrectly, that $\frac{dV}{dt} = f(5) = 6.517.$ The answer point was not earned because the student's expression for $\frac{dV}{dt}$ is not eligible to earn the answer point.

2017

AP® CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 2
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 2

(a) $\int_0^2 f(t)\,dt = 20.051175$ 20.051 pounds of bananas are removed from the display table during the first 2 hours the store is open.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $f'(7) = -8.120$ (or -8.119) After the store has been open 7 hours, the rate at which bananas are being removed from the display table is decreasing by 8.120 (or 8.119) pounds per hour per hour.	$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$
(c) $g(5) - f(5) = -2.263103 < 0$ Because $g(5) - f(5) < 0$, the number of pounds of bananas on the display table is decreasing at time $t = 5$.	$2 : \begin{cases} 1 : \text{considers } f(5) \text{ and } g(5) \\ 1 : \text{answer with reason} \end{cases}$
(d) $50 + \int_3^8 g(t)\,dt - \int_0^8 f(t)\,dt = 23.347396$ 23.347 pounds of bananas are on the display table at time $t = 8$.	$3 : \begin{cases} 2 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

$\int_0^2 f(t)\,dt = 20.051$ lbs.

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

$f'(7) = -8.119$ lbs. per hour^2

At time $t=7$, the rate at which customers remove bananas from the display table is decreasing at a rate of 8.119 lbs. per hour per hour.

Do not write beyond this border.

Continue question 2 on page 7.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

$f(5) = 13.796$
$g(5) = 11.532$

The number of pounds of bananas on the display table is decreasing at time $t=5$, because $f(5) > g(5)$.

(d) How many pounds of bananas are on the display table at time $t = 8$?

$50 - \int_0^8 f(t)\,dt + \int_3^8 g(t)\,dt = 23.347$ lbs.

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

rate at which bananas are removed

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

$$\int_0^2 \left(10 + (0.8t)\sin\left(\frac{t^3}{100}\right)\right)dt = \boxed{30.3867 \text{ bananas}}$$

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

$f(t) = 10 + 0.8t \cdot \sin\left(\frac{t^3}{100}\right)$

$f'(t) = 0 + \left[(0.8)(\sin(\frac{t^3}{100}) + (\cos\frac{t^3}{100})\left(\frac{3t^2 \cdot 100}{(100)^2}\right)(0.8t)\right]$

$f'(t) = 0.8\sin\left(\frac{t^3}{100}\right) + \frac{300t^2}{100^2}\left[\cos\left(\frac{t^3}{100}\right)\right](0.8t)$

$f'(7) = -8.11954$

The rate at which the banana pile is decreasing

Continue question 2 on page 7.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

$f(5) = 12.38$ pounds / hr

$g(5) = 11.53$ pounds / hr

The number of pounds of bananas is decreasing at $t = 5$ because the rate at which they are being removed [f(t)] is greater than the rate at which it is being added [g(t)].

(d) How many pounds of bananas are on the display table at time $t = 8$?

$$50 - \int_0^8 \left[10 + (0.8t)\sin\left(\frac{t^3}{100}\right)\right]dt + \int_3^8 \left[3 + 2.4\ln(t^2 + 2t)\right]dt$$

$$50 - 85.6167 + 58.9641 = \boxed{23.3474 \text{ pounds of bananas}}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

$= 50 - \int_0^2 f(t)\,dt$

29.9 Pounds

approximately 30 pounds of bananas are removed from the display table during the first 2 hours

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

$\frac{d}{dx}(f(x)) \Big|_{x=7}$

≈ -8.12 pounds per hour squared

f'(t) is the acceleration rate at which bananas are removed from the display table measured in pounds per hour squared

- f'(t) is the derivative of f(t)

Do not write beyond this border.

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

Continue question 2 on page 7.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

$f'(t) > 0$ therefore the number of pounds of bananas on the display is increasing at $t = 5$

$\frac{d}{dt}(f(t)) \Big|_{x=5}$
$\approx 1.7 > 0$

(d) How many pounds of bananas are on the display table at time $t = 8$?

$\int_0^8 (f(t))\, dt$
≈ 85.6
approximately, there are 86 pounds of bananas on the display table at $t=8$

Do not write beyond this border.

Question 2

Overview

The context for this problem is the removal and restocking of bananas on a display table in a grocery store during a 12-hour period. Initially, there are 50 pounds of bananas on the display table. The rate at which customers remove bananas from the table is modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \quad \text{for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. Three hours after the store opens, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened. In part (a) students were asked how many pounds of bananas are removed from the display table during the first 2 hours the store is open. Students needed to realize that the amount of bananas removed from the table during a time interval is found by integrating the rate at which bananas are removed across the time interval. Thus, students needed to express this amount as $\int_0^2 f(t)\,dt$ and use the graphing calculator to produce a numeric value for this integral. [LO 3.4E/EK 3.4E1] In part (b) students were asked to find $f'(7)$ and, using correct units, explain the meaning of $f'(7)$ in the context of the problem. Students were expected to use the graphing calculator to evaluate the derivative, and explain that the rate at which bananas are being removed from the display table 7 hours after the store has been open is decreasing by 8.120 pounds per hour per hour. [LO 2.3A/EK 2.3A1, LO 2.3D/EK 2.3D1] In part (c) students were asked to determine, with reason, whether the number of pounds of bananas on the display table is increasing or decreasing at time $t = 5$. This can be determined from the sign of the difference between the rate at which bananas are added to the table and the rate at which they are removed from the table. Thus, students needed to evaluate the difference $g(5) - f(5)$ on the graphing calculator and report that the number of pounds of bananas on the display table is decreasing because this value is negative. [LO 2.2A/EK 2.2A1] In part (d) students were asked how many pounds of bananas are on the display table at time $t = 8$. The number of pounds of bananas added to the table by time $t = 8$ is given by $\int_3^8 g(t)\,dt$, and the number of pounds of bananas removed from the table by that time is given by $\int_0^8 f(t)\,dt$. Thus, using that there were initially 50 pounds of bananas on the table, the expression $50 + \int_3^8 g(t)\,dt - \int_0^8 f(t)\,dt$ gives the number of pounds of bananas on the table at time $t = 8$. Students needed to evaluate this expression using the numeric integration capability of the graphing calculator. [LO 3.4E/EK 3.4E1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 2A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the student earned the first point for the definite integral $\int_0^2 f(t)\,dt$ and the second point for computing the correct value. In part (b) the student earned the first point for correctly computing $f'(7) = -8.119.$ The student earned the second point by having the appropriate units, mentioning the time $t = 7,$ and correctly describing the meaning in context as "the rate at which customers remove bananas from the display table." In part (c) the student earned the first point for the expressions $f(5)$ and $g(5)$ on the left-hand side of the equations in the first two lines. The student earned the second point for reaching the correct conclusion of "decreasing" by comparing these two correct values. In part (d) the student earned both integral points: 1 point for each of the definite integrals $\int_0^8 f(t)\,dt$ and $\int_3^8 g(t)\,dt.$ The student earned the third point for computing the correct value.

Sample: 2B
Score: 6

The response earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student earned the first point for the definite integral $\int_0^2 \left(10 + (0.8t)\sin\left(\frac{t^3}{100}\right)\right) dt$ on the left-hand side of the equation. The student did not earn the second point because the value produced is incorrect. In part (b) the student earned the first point in the fourth line for $f'(7) = -8.11954.$ The student did not earn the second point because units are not included, and no mention is made of the time $t = 7$ in the explanation. In part (c) the student earned the first point for the expressions $f(5)$ and $g(5)$ on the left-hand side of the equations in the first two lines. The student reports each of these values to only two decimal places, which is acceptable because this is intermediate work. However, the value for $f(5)$ is incorrect, so the student is not eligible for the second point. In part (d) the student earned both integral points: 1 point for each of the definite integrals in the first line. The student earned the third point for the expression $50 - 85.6167 + 58.9641.$ The student chooses to simplify and does so correctly because the boxed answer is accurate to at least three decimal places.

Sample: 2C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d). In part (a) the student earned the first point for the definite integral in the first line. The student did not earn the second point because the value produced is incorrect. In part (b), although the student has changed the variable to x, the student is not penalized for this and earned the first point in the second line with the value $-8.12.$ This is accurate to three decimal places because the thousandths digit is 0. The student did not earn the second point because the student does not mention the time $t = 7$ in the explanation, and the generic term "acceleration" does not capture the context of the problem. In part (c) the student did not earn the first point because only $f'(5)$ is considered. The student is not eligible for the second point. In part (d) the student earned 1 of the 2 integrals points for the definite integral in the first line. The student does not have a definite integral involving $g(t),$ so the student is not eligible for the third point.

2017

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 3
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

AP® CALCULUS AB/CALCULUS BC
2017 SCORING GUIDELINES

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

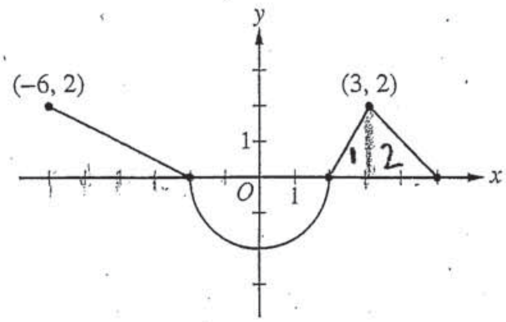
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}.$

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\ \text{with explanation} \end{cases}$

NO CALCULATOR ALLOWED

3A1 3A1



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$$f(-6) = \left(\int_{-2}^{-6} f'(x)\,dx\right) + f(-2)$$

$$f(-6) = 3$$

$$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$$

$$f(5) = 10 - 2\pi$$

(b) On what intervals is f increasing? Justify your answer.

f is increasing on $x = [-6, -2] \cup [2, 5]$, since $f' > 0$ on the interval $x \in [-6, -2) \cup (2, 5)$

Continue question 3 on page 15.

3A2

NO CALCULATOR ALLOWED

3A2

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum of f on $[-6,5]$ is $7-2\pi$, since $f(2) < f(5)$ and $f(-6)$ (the endpoints) and $f(2) < f(-2)$ the other critical point, by EVT

Endpoints

$f(-6)=3$

$f(5)=10-2\pi$

critical points

$f'=0$

$f(-2)=7$

$f(2)=7-2\pi$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$$f''(-5) = \frac{-1}{2}$$

$f''(3) = \text{DNE}$, as the

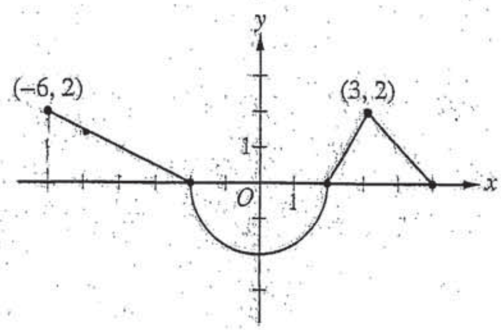
$$\lim_{x\to 3^+} \frac{f'(x)-2}{x-3} \neq \lim_{x\to 3^-} \frac{f'(x)-2}{x-3}$$

Therefore it is impossible to take a derivative at $x=3$ in f'

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx$

$f(-6) = 7 - \frac{4 \times 2}{2} = \boxed{3}$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$

$f(5) = 7 + \frac{3 \times 2}{2} - \frac{1}{2}\pi \times 2^2 = \boxed{10 - 2\pi}$

(b) On what intervals is f increasing? Justify your answer.

Since on intervals of $(-6, 2)$ and $(2, 5)$; $f'(x) > 0$

then $f(x)$ is increasing on intervals $[-6, 2]$ and $[2, 5]$

Do not write beyond this border.

Continue question 3 on page 15.

3B2

NO CALCULATOR ALLOWED

3B2

Do not write beyond this border.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

$f(x)$ has its absolute minimum on either two endpoints and where $f'(x)=0$

according to the graph: $f'(-2)=f'(2)=0$

x	$f(x)$
-6	3
-2	7
2	$7-2\pi$
5	$10-2\pi$

according to the table, $f(x)$ reaches its absolute minimum value $\boxed{7-2\pi}$ at $x=2$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(-5) = \frac{d}{dx} f'(x) \Big|_{x=-5} = \frac{2}{-4} = \boxed{-\frac{1}{2}}$

Since $\lim_{x \to 3^-} f''(x) \neq \lim_{x \to 3^+} f''(x)$

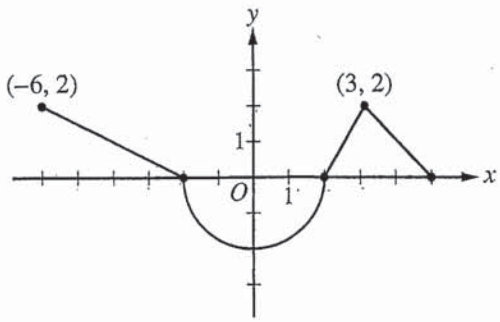
then $f(x)$ is not differentiable at $x=3$

therefore, $f''(3)$ does not exist.

3C,

NO CALCULATOR ALLOWED

3C,



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = \frac{2}{5} \times (2) = \frac{4}{5}$

$f(5) = 0$

(b) On what intervals is f increasing? Justify your answer.

From $[-6, -2]$ and $[2, 5]$, f is increasing because the graph of $f'(x)$ is > 0 from $[-6, -2)$ and $(2, 5)$

Do not write beyond this border.

Do not write beyond this border.

Continue question 3 on page 15.

3C2

NO CALCULATOR ALLOWED

3C2

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum value of 0 at $x = 2$.
because the graph of f' changes sign from negative to positive at $x = 2$.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(3)$ is an inflection point because f' increases on [2, 3] and decreases on [2, 4].
$f''(-5)$ does not exist because the graph of f' from [-6, -2] has a slope of $\frac{2}{5}$.

Do not write beyond this border.

Question 3

Overview

In this problem students were given that a function f is differentiable on the interval $[-6, 5]$ and satisfies $f(-2) = 7.$ For $-6 \le x \le 5$, the derivative of f is specified by a graph consisting of a semicircle and three line segments. In part (a) students were asked to find values of $f(-6)$ and $f(5).$ For each of these values, students needed to recognize that the net change in f, starting from the given value $f(-2) = 7,$ can be computed using a definite integral of $f'(x)$ with a lower limit of integration -2 and an upper limit the desired argument of f. These integrals can be computed using properties of the definite integral and the geometric connection to areas between the graph of $y = f'(x)$ and the x-axis. Thus, students needed to add the initial condition $f(-2) = 7$ to the values of the definite integrals for the desired values. [LO 3.2C/EK 3.2C1] In part (b) students were asked for the intervals on which f is increasing, with justification. Since f' is given on the interval $[-6, 5]$, f is differentiable, and thus also continuous, on that interval. Therefore, f is increasing on closed intervals for which $f'(x) > 0$ on the interior. Students needed to use the given graph of f' to see that $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5),$ so f is increasing on the intervals $[-6, -2]$ and $[2, 5],$ connecting their answers to the sign of f'. [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1] In part (c) students were asked for the absolute minimum value of f on the closed interval $[-6, 5],$ and to justify their answers. Students needed to use the graph of f' to identify critical points of f on the interior of the interval as $x = -2$ and $x = 2.$ Then they can compute $f(-2)$ and $f(2),$ similarly to the computations in part (a), and compare these to the values of f at the endpoints that were computed in part (a). Students needed to report the smallest of these values, $f(2) = 7 - 2\pi$ as the answer. Alternatively, students could have observed that the minimum value must occur either at a point interior to the interval at which f' transitions from negative to positive, at a left endpoint for which f' is positive immediately to the right, or at a right endpoint for which f' is negative immediately to the left. This reduces the options to $f(-6) = 3$ and $f(2) = 7 - 2\pi.$ [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1, LO 3.3A/EK 3.3A3] In part (d) students were asked to determine values of $f''(-5)$ and $f''(3),$ or to explain why the requested value does not exist. Students needed to find the value $f''(-5)$ as the slope of the line segment on the graph of f' through the point corresponding to $x = -5.$ The point on the graph of f' corresponding to $x = 3$ is the juncture of a line segment of slope 2 on the left with one of slope -1 on the right. Thus, students needed to report that $f''(3)$ does not exist, and explain why the given graph of f' shows that f' is not differentiable at $x = 3.$ Student explanations could be done by noting that the left-hand and right-hand limits at $x = 3$ of the difference quotient $\frac{f'(x) - f'(3)}{x - 3}$ have differing values (2 and $-1,$ respectively), or by a clear description of the relevant features of the graph of f' near $x = 3.$ [LO 1.1A(b)/EK 1.1A3] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 3A
Score: 9

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-2}^{-6} f'(x)\,dx$ to find $f(-6) = 3.$ Thus, the student earned the first and second points. The student uses $f(-2)$ again with an

Question 3 (continued)

appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, where f is increasing. The student justifies the intervals with a discussion of $f' > 0$ for $[-6, -2)$ and $(2, 5)$. The student earned both points. In part (c) the student considers $x = -6, -2, 2,$ and 5 as potential locations for the absolute minimum value. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student finds $f''(-5) = -\frac{1}{2}$ and earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$ to explain why the derivative of $f'(x)$ does not exist and earned the second point.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-6}^{-2} f'(x)\,dx$ to find $f(-6) = 3$. Thus, the student earned the first and second points. The student uses $f(-2)$ again with an appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student presents two intervals, $[-6, 2)$ and $(2, 5)$. Because $f'(x) < 0$ on $(-2, 2)$, f is decreasing on $[-2, 2]$. The student is not eligible to earn any points because of the presence of an interval containing points where $f'(x) < 0$. Thus, the student did not earn any points. In part (c) the student investigates where $f'(x) = 0$ and identifies $f'(-2)$ and $f'(2)$. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student identifies $f''(-5)$ as the derivative of $f'(x)$ at $x = -5$ and finds $f''(-5) = -\frac{1}{2}$. The student earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$. The student states that "$f(x)$ is not differentiable at $x = 3$," which contradicts the given statement in the problem that f is differentiable on the closed interval $[-6, 5]$. The student did not earn the second point.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student never uses the initial condition, incorrectly evaluates $f(-6)$ as $\frac{4}{5}$, and incorrectly evaluates $f(5)$ as 0. The student earned no points. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, on which f is increasing. The student justifies the intervals with "$f'(x)$ is > 0 from $[-6, -2)$ and $(2, 5)$." The student earned both points. In part (c) the student considers $x = 2$ and earned the first point. The student presents an incorrect answer for the absolute minimum value with an incorrect justification. The student does not evaluate $f(x)$ at the critical values and endpoints in order to determine the

Question 3 (continued)

absolute minimum value. The student did not earn the second point. In part (d) the student incorrectly determines that $f''(-5)$ has a value of $\frac{2}{5}$ and did not earn the first point. The student states that "$f''(3)$ is an inflection point" and does not state that $f''(3)$ does not exist. The student did not earn the second point.

2017

AP® CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 4
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$ $H(0) = 91$ An equation for the tangent line is $y = 91 - 16t$. The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.	$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$
(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$ $H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$ Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.	1 : underestimate with reason
(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$ $\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$ $3(G - 27)^{1/3} = -t + C$ $3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$ $3(G - 27)^{1/3} = 12 - t$ $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$ The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.	$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

4A₁

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\left.\frac{dH}{dt}\right|_{H=91} = -\frac{1}{4}(91-27) = \left(-\frac{1}{4}\right)(64) = -16$

$\begin{array}{r} 8\ 11 \\ 27 \\ \hline 64 \end{array}$

$A(t) = -16(t-0) + 91$

$\boxed{A(t) = -16t + 91}$

$A(3) = -16(3) + 91$
$= -48 + 91$
$= \boxed{43°C \text{ at } t = 3 \text{ minutes}}$

$\begin{array}{r} 8\ 11 \\ 9\cancel{1} \\ -48 \\ \hline 43 \end{array}$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(H-27)$

$\frac{d^2H}{dt^2} = -\frac{1}{4}\left(\frac{dH}{dt}\right)$

$= -\frac{1}{4}\left(-\frac{1}{4}\right)(H-27)$

$\frac{d^2H}{dt^2} = \frac{1}{16}(H-27)$

$H > 27$ for all $t > 0$

$\frac{d^2H}{dt^2}$ is always positive, so part (a) is an

$\boxed{\text{underestimate.}}$

Continue question 4 on page 17.

Do not write beyond this border.

4A$_2$

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$\int \frac{dG}{(G-27)^{2/3}} = \int -1\,dt$$

$$\int (G-27)^{-2/3}\,dG = -t + C$$

$$3(G-27)^{1/3} = -t + C_1$$

$$(G-27)^{1/3} = -\frac{t}{3} + C_2$$

$$G - 27 = \left(-\frac{t}{3} + C_2\right)^3$$

$$G = \left(-\frac{t}{3} + C_2\right)^3 + 27$$

at $(0, 91)$

$$91 = (0 + C_2)^3 + 27$$

$$91 = (C_2)^3 + 27$$

$$64 = (C_2)^3 \rightarrow C_2 = 4$$

91
27

$$\boxed{G(t) = \left(-\frac{t}{3} + 4\right)^3 + 27}$$

$$G(3) = \left(-\frac{3}{3} + 4\right)^3 + 27$$

$$= (3)^3 + 27$$

$$= 27 + 27$$

$$\boxed{G(3) = 54°C}$$

Do not write beyond this border.

NO CALCULATOR ALLOWED

4B1

Do not write beyond this border.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(91-27)$

$\frac{dH}{dt} = -\frac{1}{4}(64)$

$\frac{dH}{dt} = -16$

eavation of tangent line:

$y - 91 = -16t$

$y = -16t + 91$

approximation at time $t=3$:

$y = -16(3) + 91$

$y = -48 + 91$

$= 43°C$ at time $t=3$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH'}{dt} = -\frac{1}{4}(H-27)$

$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$

$\frac{d^2H}{dt^2} = -\frac{1}{4}$

underestmate because the value is less that the estimated value of $\frac{d^2H}{dt^2}$ at $t=3$

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

Continue question 4 on page 17.

4B2

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Do not write beyond this border.

$$\frac{1}{-(G-27)^{2/3}} dG = 1\, dt$$

$$3(G-27)^{1/3} = t + C$$

$$3(91-27)^{1/3} = 0 + C$$

$$3(64)^{1/3} = C$$

$$3(4) = C$$

$$12 = C$$

$$3(G-27)^{1/3} = t + 12$$

$$(G-27)^{\cancel{1/3}} = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(t) = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(3) = \left(\frac{15}{3}\right)^3 + 27$$

$$G(3) = (5)^3 + 27$$

$$G(3) = 125 + 27$$

$$G(3) = 152$$

152°C at time $t = 3$

GO ON TO THE NEXT PAGE.

4C1

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(91-27)$

$= -\frac{1}{4}(64) = -16$

$H - 91 = -16(t)$

$H = -16t + 91$

$H = -16(3) + 91$

$H = -48 + 91$

$H = 43°C$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$

$\frac{d^2H}{dt^2} = -\frac{1}{4}$

This is an underestimate because $\frac{dH}{dt}$ + $\frac{d^2H}{dt^2}$ are both decreasing

Continue question 4 on page 17.

4C2

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$\frac{dG}{dt} = -(G-27)^{\frac{2}{3}}$

$\int \frac{dG}{dt} = \int -(G-27)^{\frac{2}{3}} dt$

$G(t) = -\int (G-27)^{\frac{2}{3}} dt$ $\quad u = G-27$

$du = dt$

$G(t) = -\int u^{\frac{2}{3}} du$

$G(t) = -\frac{u^{\frac{5}{3}}}{\frac{5}{3}}$ $\quad -\frac{3}{5}(G-27)^{\frac{5}{3}}$

$-\frac{3}{5}(91-27)^{\frac{5}{3}}$

$-\frac{3}{5}(64)^{\frac{5}{3}}$

64
$\times 4$
256
$\times 4$
1024

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 4

Overview

The context for this problem is the internal temperature of a boiled potato that is left to cool in a kitchen. Initially at time $t = 0,$ the potato's internal temperature is 91 degrees Celsius, and it is given that the internal temperature of the potato exceeds 27 degrees Celsius for all times $t > 0$. The internal temperature of the potato at time t minutes is modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27),$ where $H(t)$ is measured in degrees Celsius and $H(0) = 91.$ In part (a) students were asked for an equation of the line tangent to the graph of H at $t = 0,$ and to use this equation to approximate the internal temperature of the potato at time $t = 3.$ Using the initial value and the differential equation, students needed to find the slope of the tangent line to be $H'(0) = -\frac{1}{4}(91 - 27) = -16$ and report the equation of the tangent line to be $y = 91 - 16t.$ Students needed to find the approximate temperature of the potato at $t = 3$ to be $91 - 16 \cdot 3 = 43$ degrees Celsius. [LO 2.3B/EK 2.3B2] In part (b) students were asked to use $\frac{d^2H}{dt^2}$ to determine whether the approximation in part (a) is an underestimate or overestimate for the potato's internal temperature at time $t = 3.$ Students needed to use the given differential equation to calculate $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H - 27).$ Then using the given information that the temperature always exceeds 27 degrees Celsius, students needed to conclude that $\frac{d^2H}{dt^2} > 0$ for all times $t.$ Thus, the graph of H is concave up, and the line tangent to the graph of H at $t = 0$ lies below the graph of H (except at the point of tangency), so the approximation found in part (a) is an underestimate. [LO 2.1D/EK 2.1D1, LO 2.2A/EK 2.2A1] In part (c) an alternate model, G, is proposed for the internal temperature of the potato at times $t < 10.$ $G(t)$ is measured in degrees Celsius and satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$ with $G(0) = 91.$ Students were asked to find an expression for $G(t)$ and to find the internal temperature of the potato at time $t = 3$ based on this model. Students needed to employ the method of separation of variables, using the initial condition $G(0) = 91$ to resolve the constant of integration, and arrive at the particular solution $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$. Students should then have reported that the model gives an internal temperature of $G(3) = 54$ degrees Celsius for the potato at time $t = 3.$ [LO 3.5A/EK 3.5A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 4A
Score: 9

The response earned all 9 points: 3 points in part (a), 1 point in part (b), and 5 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91 - 27).$ The second point was earned for the tangent line $A(t) = -16(t - 0) + 91.$ Note that the student names the tangent line $A(t)$ and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student has the correct answer of "underestimate" and supports the answer with correct reasoning. The student has the correct second derivative in line 2 on the left and states that the second derivative is always positive in line 2 on the right. The

Question 4 (continued)

student earned the point. In part (c) the student earned the first point for separation in line 2. The antiderivatives are correct in line 4 and earned the second point. The third point was earned for the constant of integration and use of the initial condition in lines 8 and 9. The fourth point was earned in line 12 for an equation involving G and t and a correct numerical value for C. As an aside, the fourth point could have been earned with an implicit equation such as $(G - 27)^{1/3} = \frac{12 - t}{3}$. The fifth point was earned for $G(t) = \left(-\frac{t}{3} + 4\right)^3 + 27$ in line 12 along with 54 in line 4 on the right. Any of the three numerical expressions in lines 1, 2, and 3 on the right together with $G(t) = \left(-\frac{t}{3} + 4\right)^3 + 27$ in line 12 would have earned the fifth point. The student chooses to simplify and does so correctly.

Sample: 4B
Score: 6

The response earned 6 points: 3 points in part (a), no point in part (b), and 3 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91 - 27)$. The second point was earned for the tangent line $y - 91 = -16t$. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative in line 3 on the left. The student did not earn the point. In part (c) the student earned the first point for separation in line 1. In line 2 the student drops a negative sign, so the second point for correct antiderivatives was not earned. Line 2 should have been: $-3(G - 27)^{1/3} = t + C$. The student's equation involving antiderivatives is eligible for the remaining 3 points. In line 3 the third point was earned for the constant of integration and use of the initial condition. In line 7 the fourth point was earned for an equation involving G and t together with the consistent numerical value for C. The fifth point was not earned because, although the answer is consistent with the work, the student's value of $G(3)$ is out of the context of the problem because $152 > 91$.

Sample: 4C
Score: 3

The response earned 3 points: 3 points in part (a), no point in part (b), and no points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91 - 27)$ in line 1 on the left. The second point was earned for the tangent line $H - 91 = -16(t)$. Note that the student uses H in place of y in the tangent line and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative of $-\frac{1}{4}$. The student did not earn the point. In part (c) there is no separation of variables, and thus, no points were earned.

2017

AP® CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 5
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 5

(a) $x_P'(t) = \dfrac{2t-2}{t^2-2t+10} = \dfrac{2(t-1)}{t^2-2t+10}$

$t^2 - 2t + 10 > 0$ for all t.

$x_P'(t) = 0 \Rightarrow t = 1$

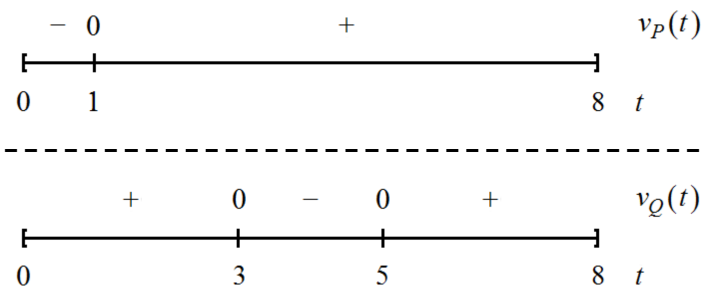
$x_P'(t) < 0$ for $0 \le t < 1$.

Therefore, the particle is moving to the left for $0 \le t < 1$.

$2 : \begin{cases} 1 : x_P'(t) \\ 1 : \text{interval} \end{cases}$

(b) $v_Q(t) = (t-5)(t-3)$

$v_Q(t) = 0 \Rightarrow t = 3, t = 5$



Both particles move in the same direction for $1 < t < 3$ and $5 < t \le 8$ since $v_P(t) = x_P'(t)$ and $v_Q(t)$ have the same sign on these intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{analysis using } v_P(t) \text{ and } v_Q(t) \end{cases}$

Note: $1/2$ if only one interval with analysis

Note: $0/2$ if no analysis

(c) $a_Q(t) = v_Q'(t) = 2t - 8$

$a_Q(2) = 2 \cdot 2 - 8 = -4$

$a_Q(2) < 0$ and $v_Q(2) = 3 > 0$

At time $t = 2$, the speed of the particle is decreasing because velocity and acceleration have opposite signs.

$2 : \begin{cases} 1 : a_Q(2) \\ 1 : \text{speed decreasing with reason} \end{cases}$

(d) Particle Q first changes direction at time $t = 3$.

$$x_Q(3) = x_Q(0) + \int_0^3 v_Q(t)\,dt = 5 + \int_0^3 \left(t^2 - 8t + 15\right) dt$$

$$= 5 + \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_{t=0}^{t=3} = 5 + (9 - 36 + 45) = 23$$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$v_p(t)$ is neg.

$$v_p(t) = x_p'(t) = \frac{1}{t^2-2t+10} \cdot (2t-2)$$

$$0 = \frac{2t-2}{t^2-2t+10}$$

$2t-2=0$

$t=1$

	$t<1$	$t>1$
$2t-2$	$-$	$+$
$t^2-2t+10$	$+$	$+$
$v_p(t)$	$-$	$+$

Particle P is moving to the left on $0 \le t < 1$.

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

$t^2 - 8t + 15 = 0$

$(t-3)(t-5) = 0$

$t = 3, 5$

	$t<3$	$3<t<5$	$t>5$
$t-3$	$-$	$+$	$+$
$t-5$	$-$	$-$	$+$
$v_Q(t)$	$+$	$-$	$+$

	$t<1$	$1<t<3$	$3<t<5$	$t>5$
$v_p(t)$	$-$	$+$	$+$	$+$
$v_Q(t)$	$+$	$+$	$-$	$+$

Particles P and Q are moving in the same direction when $1<t<3$ and $5<t\le 8$.

Do not write beyond this border.

Continue question 5 on page 19.

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$a_Q(t) = 2t - 8$

$a_Q(2) = 2(2) - 8 = -4$

The speed of particle Q is decreasing because the particle has a positive velocity but negative acceleration when $t = 2$.

(d) Find the position of particle Q the first time it changes direction.

@ $t = 3$, particle Q first changes direction

$x_Q(t) = \int (t^2 - 8t + 15)\,dt = \frac{t^3}{3} - 4t^2 + 15t + C$

$x_Q(0) = C = 5 \Rightarrow x_Q(t) = \frac{t^3}{3} - 4t^2 + 15t + 5$

$x_Q(3) = \frac{3^3}{3} - 4(3)^2 + 15(3) + 5 = 9 - 36 + 45 + 5 = \boxed{23}$

Do not write beyond this border.

NO CALCULATOR ALLOWED

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$V_P(t) = \frac{1}{t^2-2t+10}(2t-2)$

$\frac{2t-2}{t^2-2t+10} < 0$

$2t-2=0$

$2(t-1)=0$

$t=1$

$V_P(0.1) = \frac{0.2-2}{0.01-0.2+10} =$ negative

$V_P(5) = \frac{8}{25-10+10} =$ positive

In the interval $0 \le t < 1$ particle P moving to the left

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

$V_Q(t) = 0$

$t = 5, 3$

$V_Q(2) = 4 - 16 + 15 =$ positive

$V_Q(4) = 16 - 24 + 15 =$ positive

$1 < t < 5$ they travel in the same direction

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$A_Q(t) = 2t - 8$

$A_Q(2) = -4$

The speed of particle Q at t=2 is decreasing since the acceleration at t=2 is negative

(d) Find the position of particle Q the first time it changes direction.

$V_Q(t) = 0$

$t^2 - 8t + 15 = 0$

$(t-5)(t-3) = 0$

$t = 5, 3$

At t=3, t first changes direction

$\int_0^3 (t^2 - 8t + 15)\,dt = x_Q(3) - x_Q(0)$

$\left(\frac{t^3}{3} - 4t^2 + 15t\right)\Big|_0^3 = x_Q(3) - 5$

$9 - 36 + 45 = x_Q(3) - 5$

$18 + 5 = 23$

$x_Q(3) = 23$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by
$s(t)$
$x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$.

Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$v_p(t) = \frac{1}{t^2-2t+10}(2t-2)$

$v_p(t) = \frac{2t-2}{t^2-2t+10}$

$0 = \frac{2t-2}{t^2-2t+10}$

$t^2-2t+10 = 2t-2$

$t^2-4t+12=0$

$\left[\ln(t^2-2t+10)\right]_0^8$

$\ln(58) - \ln(10)$

$\ln\left(\frac{58}{10}\right)$

@ $x = \ln\left(\frac{58}{10}\right)$

$\left[\frac{2t-2}{t^2-2t+10}\right]_0^8$

$\frac{7}{29} - -\frac{1}{5}$

Do not write beyond this border.

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

@ $t = 5, 2$, and -1

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$a(t) = 2t - 8$

$a(2) = 2(2) - 8$

$a(2) = -4$

the speed of the particle is decreasing because the acceleration is negative and the velocity is positive @ $t=2$

$v(2) = 2^2 - 8(2) + 15$

$v(2) = 4 - 16 + 15$

$= 3$

(d) Find the position of particle Q the first time it changes direction.

$0 = t^2 - 8t + 15$

$(t-5)(t-3) = 0$

$t = 5, t = 3$

$\boxed{\text{at } t = 5}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem, two particles, P and Q, are moving along the x-axis. For $0 \le t \le 8$, the position of particle P is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while particle Q has position 5 at $t = 0$ and velocity $v_Q(t) = t^2 - 8t + 15$. In part (a) students were asked for those times t, $0 \le t \le 8$, when particle P's motion is to the left. Using the chain rule, students needed to find an expression for the velocity of particle P at time t by differentiating the position $x_P(t)$. By analyzing the sign of this derivative to determine those times t with $x'_P(t) < 0$, students should have concluded that particle P is moving left for $0 \le t < 1$. [LO 2.1C/EK 2.1C2-2.1C4, LO 2.3C/EK 2.3C1] In part (b) students were asked for all times t, $0 \le t \le 8$, during which both particles travel in the same direction. Using the velocity of particle P, $v_P(t) = x'_P(t)$, found in part (a), and the given velocity $v_Q(t)$ for particle Q, students needed to find those subintervals of $0 \le t \le 8$ on which both $v_P(t)$ and $v_Q(t)$ have the same sign. Students should have responded that for $1 < t < 3$ and for $5 < t \le 8$, noting that both $v_P(t)$ and $v_Q(t)$ are positive on these intervals, both particles travel in the same direction (to the right). There is no time when both velocities are negative. [LO 2.3C/EK 2.3C1] In part (c) students were asked for the acceleration of particle Q at time $t = 2$, and to determine, with explanation, whether particle Q's speed is increasing, decreasing, or neither at time $t = 2$. Students needed to differentiate $v_Q(t)$ to find that the acceleration of particle Q is given by $a_Q(t) = v'_Q(t) = 2t - 8$, and report that particle Q's acceleration at time $t = 2$ is $a_Q(2) = -4$. Students should have explained that particle Q's speed is decreasing at time $t = 2$ because the velocity and acceleration of particle Q have opposite signs at that time. [LO 2.1C/EK 2.1C2, LO 2.3C/EK 2.3C1] In part (d) students were asked to find the position of particle Q the first time it changes direction. Using the analysis of the sign of $v_Q(t)$ done in part (b), students should have concluded that the first change of direction of particle Q's motion occurs at time $t = 3$. The net change in position of particle Q across the time interval $[0, 3]$ is given by $\int_0^3 v_Q(t)\,dt$. Students needed to evaluate this integral using the Fundamental Theorem of Calculus and use the initial position of particle Q to find that particle Q's position at time $t = 3$ is $5 + \int_0^3 v_Q(t)\,dt = 23$. [LO 3.3B(b)/EK 3.3B2, LO 3.4C/EK 3.4C1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the student earned the first point with a correct derivative expression $x'_P(t) = \dfrac{1}{t^2 - 2t + 10} \cdot (2t - 2)$. The student earned the second point in the last sentence when the student writes "Particle P is moving to the left on $0 \le t < 1$." In part (b) the student earned the first point by identifying that "Particles P and Q are moving in the same direction when $1 < t < 3$ and $5 < t \le 8$" with some analysis. The student earned the second point by identifying all of the following connections between the sign of the velocity and its associated interval: $v_P(t) > 0$ for $(1, 8]$, $v_P(t) < 0$ for $[0, 1)$, $v_Q(t) < 0$ for $(3, 5)$, and $v_Q(t) > 0$ for $[0, 3)$ and $(5, 8]$. This information is included on labeled sign charts. In part (c) $a_Q(2) = 2(2) - 8$ in the second

Question 5 (continued)

line would earn the first point without simplification. The student chooses to simplify, does so correctly, and earned the first point. The student earned the second point with a correct conclusion that speed is decreasing and the reasoning using the sign of the acceleration and the sign of the velocity at $t = 2.$ In part (d) the student earned the antiderivative point in the second line with the expression $\frac{t^3}{3} - 4t^2 + 15t + C.$ The student earned the initial condition point with $x_Q(t) = \frac{t^3}{3} - 4t^2 + 15t + 5$ in the third line. $\frac{3^3}{3} - 4(3)^2 + 15(3) + 5$ or $9 - 36 + 45 + 5$ would have earned the answer point without simplification. The student chooses to simplify and does so correctly, so the student earned the answer point.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student earned the first point in the first line with a correct derivative expression $v_P(t) = \frac{1}{t^2 - 2t + 10}(2t - 2).$ The student earned the second point by "In the interval $0 \le t < 1$ particle P moving to the left." In part (b) the student's interval is incorrect, so the first point was not earned. Without correct intervals, the student is not eligible for the second point. In part (c) the student earned the first point with $A_Q(2) = -4$ in the second line. Although the student wrote that the speed of particle Q is decreasing, the student's reason is incorrect because the student does not use both the sign of the acceleration and the sign of the velocity of particle Q at $t = 2.$ The student did not earn the second point. In part (d) the student earned the initial condition point with the equation $\int_0^3 (t^2 - 8t + 15)\, dt = x_Q(3) - x_Q(0).$ The student earned the antiderivative point in the next line with the expression $\left(\frac{t^3}{3} - 4t^2 + 15t\right).$ The numeric expression $18 + 5$ would have earned the answer point without simplification. The student chooses to simplify and does so correctly, so the student earned the answer point.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student earned the first point in the first line with a correct derivative expression $v_P(t) = \frac{1}{t^2 - 2t + 10}(2t - 2).$ The student identifies no interval on which particle P is moving left. The student did not earn the second point. In part (b) the student did not earn the first point because the student does not include any intervals. Without correct intervals, the student is not eligible for the second point. In part (c) $a(2) = 2(2) - 8$ would have earned the first point without simplification. The student chooses to simplify and does so correctly, so the student earned the first point. The student earned the second point by identifying that the speed of particle Q is decreasing and reasoning using the sign of the acceleration and the sign of the velocity at $t = 2.$ In part (d) the student does not find the antiderivative, use the initial condition, or include an answer. Thus, the student earned no points.

2017

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

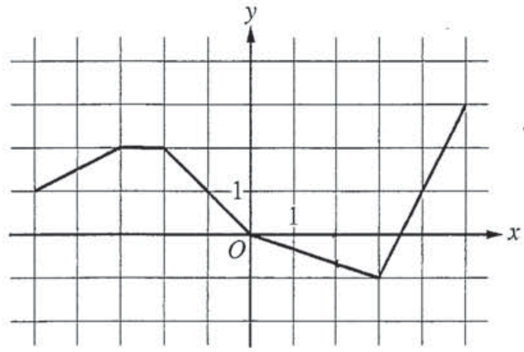
Inside:

- ☑ Free Response Question 6
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 6

(a) $f'(x) = -2\sin(2x) + \cos x \, e^{\sin x}$ $f'(\pi) = -2\sin(2\pi) + \cos \pi \, e^{\sin \pi} = -1$	$2 : f'(\pi)$
(b) $k'(x) = h'(f(x)) \cdot f'(x)$ $k'(\pi) = h'(f(\pi)) \cdot f'(\pi) = h'(2) \cdot (-1)$ $= \left(-\frac{1}{3}\right)(-1) = \frac{1}{3}$	$2 : \begin{cases} 1 : k'(x) \\ 1 : k'(\pi) \end{cases}$
(c) $m'(x) = -2g'(-2x) \cdot h(x) + g(-2x) \cdot h'(x)$ $m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2)$ $= -2(-1)\left(-\frac{2}{3}\right) + 5\left(-\frac{1}{3}\right) = -3$	$3 : \begin{cases} 2 : m'(x) \\ 1 : m'(2) \end{cases}$
(d) g is differentiable. $\Rightarrow$ g is continuous on the interval $[-5, -3]$. $\frac{g(-3) - g(-5)}{-3 - (-5)} = \frac{2 - 10}{2} = -4$ Therefore, by the Mean Value Theorem, there is at least one value c, $-5 < c < -3$, such that $g'(c) = -4$.	$2 : \begin{cases} 1 : \frac{g(-3) - g(-5)}{-3 - (-5)} \\ 1 : \text{justification, using Mean Value Theorem} \end{cases}$

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

$m = -\frac{1}{3}x$

$-2/3$

$\frac{-1-0}{3-0} = -\frac{1}{3}$

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$(-1, 0)$ $(1, 0)$

$f'(x) = -2\sin(2x) + e^{\sin x}(\cos x)$

$f'(\pi) = -2\sin(2\pi) + e^{\sin \pi}(\cos \pi)$

$= 0 + 1(-1)$

$= \underline{-1}$

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$\cos(2\pi) + e^{\sin \pi}$

$1 + 1$

$k'(x) = h'(f(x))(f'(x))$

$k'(\pi) = h'(f(\pi))\, f'(\pi)$

$= h'(2)(-1)$

$= 1/3$

Do not write beyond this border.

Continue question 6 on page 21.

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$$m'(x) = [g'(-2x)(-2) \cdot h(x)] + [h'(x) \cdot g(-2x)]$$
$$m'(2) = [2g'(-4) \cdot h(2)] + [h'(2) \cdot g(-4)]$$
$$= (2 \cdot -2/3) + (-1/3 \cdot 5)$$
$$-\frac{4}{3} + -\frac{5}{3}$$
$$= -\frac{9}{3}$$
$$= \underline{-3}$$

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

$g(x)$ = differentiable $\rightarrow$ $g(x)$ continuous $\therefore$
Mean value theorem states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

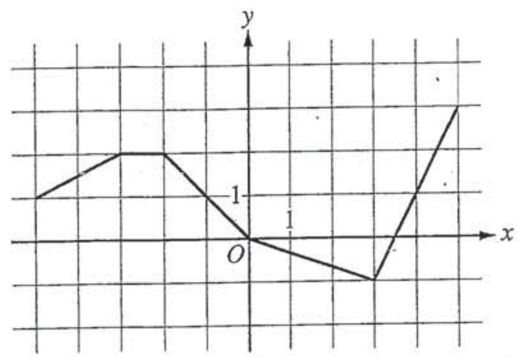
$$\frac{g(-3) - g(-5)}{-3 + 5} = \frac{2 - 10}{2}$$
$$= \frac{-8}{2}$$
$$= -4$$

yes, there's a number c in the closed interval such that $g'(c) = -4$.

Do not write beyond this border.

NO CALCULATOR ALLOWED

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$f(x) = \cos(2x) + e^{\sin x}$

$f'(x) = -2\sin(2x) + (\cos x)e^{\sin x}$

$f'(\pi) = -2\sin(2\pi) + (\cos\pi)e^{\sin\pi}$

$f'(\pi) = -2(0) + (-1)e^{(0)}$

$f'(\pi) = 0 + -1(1) = -1$

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$k(x) = h(f(x))$

$k'(x) = h'(f(x))\, f'(x)$

$k'(\pi) = h'(f(\pi))\, f'(x)$

$k'(\pi) = [h'(2)](-1)$

$k'(\pi) = (-\frac{1}{3})(-1)$

$f(\pi) = \cos(2\pi) + e^{\sin\pi}$

$f(\pi) = 1 + e^{0} = 1 + 1 = 2$

Do not write beyond this border.

Continue question 6 on page 21.

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$m(x) = g(-2x) \cdot h(x)$

$m'(x) = -2g'(-2x) \cdot h(x) + h'(x) \cdot g(-2x)$

$m'(2) = -2\, g'(-4) \cdot h(2) + h'(2) \cdot g(-4)$

$m'(2) = -2[-1] \cdot \frac{2}{3} + \left(\frac{1}{3}\right) \cdot 5$

$m'(2) = \frac{8}{3} + \frac{5}{3} = \frac{13}{3}$

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

No, there isn't a number c in the interval [-5,-3] such that $g'(c) = 4$ because the values of $g'(x)$ are increasing from [-5,-3] and show no sign that the values will decreasing. This can be justified by the midpoint theorem as

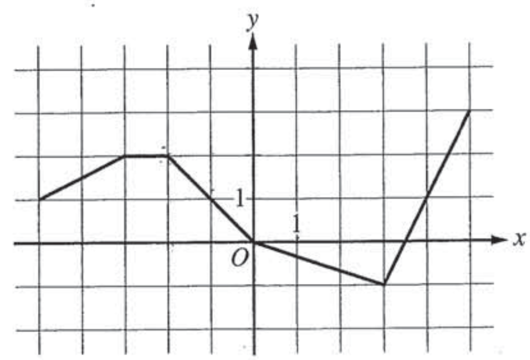
x	$g'(x)$
-5	-3
-4	1
-3	4
c	$-3 \le x \le 4$

Do not write beyond this border.

6C1 6C1

NO CALCULATOR ALLOWED

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$$f'(x) = -2\sin(2x) + e^{\sin x}(\cos(x))$$

$$f'(\pi) = -2\sin(2\pi) + e^{\sin\pi}\cos(\pi)$$

$$0 + 1(-1)$$

$$f'(\pi) = \boxed{-1}$$

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$$k'(x) = h'(f(x))f'(x)$$

$$k'(\pi) = h'(f(\pi))f'(\pi)$$

Continue question 6 on page 21.

6C2 6C2

NO CALCULATOR ALLOWED

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$m'(x) = g'(-2x)(-2) \cdot h'(x)$

$m'(2) = g'(-4) -2 \cdot h'(-2)$

$m'(2) = (4 -2) \cdot -1$

$m'(2) = 2 \cdot -1$

$\boxed{m'(2) = -2}$

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

NO, because there are no points in that interval with a slope of -4.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 6

Overview

This problem deals with multiple functions. Function f is defined by $f(x) = \cos(2x) + e^{\sin x}$. Function g is differentiable and values of $g(x)$ and $g'(x)$ corresponding to integer values of x from $x = -5$ to $x = 0$, inclusive, are given in a table. Function h is defined on $[-5, 5]$ and the graph of h, comprised of five line segments, is given. In part (a) students were asked for the slope of the line tangent to the graph of f at $x = \pi$. Using the sum and chain rules for differentiation and the derivatives of trigonometric and exponential functions to differentiate $f(x)$, students needed to evaluate $f'(\pi)$ to find the slope of the tangent line. [LO 2.1C/EK 2.1C2-2.1C4, LO 2.3B/EK 2.3B1] In part (b) the function k is defined by $k(x) = h(f(x))$, and students were asked to find $k'(\pi)$. Students needed to apply the chain rule and determine the value of $h'(2)$ from the graph of h to arrive at the value for $k'(\pi)$. [LO 2.1C/EK 2.1C4, LO 2.2A/EK 2.2A2] In part (c) the function m is defined by $m(x) = g(-2x) \cdot h(x)$, and students were asked to find $m'(2)$. Students needed to apply the product and chain rules for differentiation, find values for $g(-4)$ and $g'(-4)$ in the table for g, and use the graph of h to determine $h(2)$ and $h'(2)$, to find $m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2) = -3$. [LO 2.1C/EK 2.1C3-2.1C4, LO 2.2A/EK 2.2A2, LO 2.3B/EK 2.3B1] In part (d) students were asked to determine whether there is a number c in the interval $[-5, -3]$ such that $g'(c) = -4$, and to justify their answers. Using the table for g, students should have confirmed that $\dfrac{g(-3) - g(-5)}{-3 - (-5)} = -4$. Given that g is differentiable, students should have concluded that g is continuous on $[-5, -3]$ and, thus, recognize that the hypotheses for the Mean Value Theorem are satisfied, and answered in the affirmative that a number c exists in the interval $[-5, -3]$ such that $g'(c) = -4$. [LO 1.2B/EK 1.2B1, LO 2.4A/EK 2.4A1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 6A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the student presents a correct expression for $f'(x)$ in line 1 and earned the points for $f'(\pi)$ in line 4. The expression in line 2 would have also earned the points for $f'(\pi)$ without simplification. The student chooses to simplify and does so correctly. Both points were earned. In part (b) the student earned the point for $k'(x)$ in line 1 and earned the point for $k'(\pi)$ in line 4. In part (c) the student earned both points for $m'(x)$ in line 1. The student would have earned the point for $m'(2)$ in line 3. The student chooses to simplify and does so correctly, so the student earned the point. In part (d) the student earned the point for the difference quotient with the statement $\dfrac{g(-3) - g(-5)}{-3 + 5}$. The student confirms the conditions for the Mean Value Theorem in the first line, goes on to connect the difference quotient with the value -4, and draws the appropriate conclusion for the Mean Value Theorem. The student earned the justification point.

Question 6 (continued)

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student presents a correct expression for $f'(x)$ in line 2 and earned the points for $f'(\pi)$ in line 5. The expression in line 3 would have also earned the points for $f'(\pi)$ without simplification. The student chooses to simplify and does so correctly. Both points were earned. In part (b) the student earned the point for $k'(x)$ in line 2 on the left and earned the point for $k'(\pi)$ in the last line on the left, after correctly evaluating and using $h'(2)$. In part (c) the student earned both points for $m'(x)$ in line 2. The student is at first eligible to earn the point for $m'(2)$ in line 4, but the student makes errors in evaluating $h(2)$ and $h'(2)$. The student did not earn the third point. In part (d) the student does not present a difference quotient and never engages with the Mean Value Theorem. As a result, the student did not earn any points.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student presents a correct expression for $f'(x)$ in line 1 and earned the points for $f'(\pi)$ in line 4. The expression in line 2 would have also earned the points for $f'(\pi)$ without simplification. The student chooses to simplify and does so correctly. Both points were earned. In part (b) the student earned the point for $k'(x)$ in line 1. The student substitutes π for x in line 2 but does not evaluate the expression. The student did not earn the point for $k'(\pi)$. In part (c) the student does not present an expression that uses the product rule. Thus, the student is not eligible to earn any points. In part (d) the student does not present a difference quotient, so that point was not earned. Because the student's conclusion is incorrect, the student is not eligible for the justification point.